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INTEGRATION BY ELLIPTIC INTEGRALS.

By **GEORGE B. McCLELLAN ZERR, A. M., Ph. D.**, Professor of Mathematics and Science, Chester High School, Chester, Pa.

We will first expand the general expression

$$(1 + e^2 - 2e \cos \varphi)^{-\frac{1}{2}(2m+1)}.$$

$$\text{Let } 2 \cos \varphi = s + s^{-1}. \quad \therefore (1 + e^2 - 2e \cos \varphi)^{-\frac{1}{2}(2m+1)} = (1 + e^2 - es - es^{-1})^{-\frac{1}{2}(2m+1)}$$

$$\begin{aligned} &= (1 - es)^{-\frac{1}{2}(2m+1)} (1 - es^{-1})^{-\frac{1}{2}(2m+1)} = \left(1 + \frac{2m+1}{2} \cdot es + \frac{(2m+1)(2m+3)}{2^2 \cdot 2!} \cdot e^2 s^2 \right. \\ &+ \frac{(2m+1)(2m+3)(2m+5)}{2^3 \cdot 3!} \cdot e^3 s^3 + \frac{(2m+1)(2m+3)(2m+5)(2m+7)}{2^4 \cdot 4!} \cdot e^4 s^4 + \dots \left. \right) \\ &\left(1 + \frac{2m+1}{2} \cdot es^{-1} + \frac{(2m+1)(2m+3)}{2^2 \cdot 2!} \cdot e^2 s^{-2} + \frac{(2m+1)(2m+3)(2m+5)}{2^3 \cdot 3!} \cdot e^3 s^{-3} \right. \\ &+ \left. \frac{(2m+1)(2m+3)(2m+5)(2m+7)}{2^4 \cdot 4!} \cdot e^4 s^{-4} + \dots \right) = \left[1 + \left(\frac{2m+1}{2} \cdot e \right)^2 \right. \\ &+ \left. \left(\frac{2m+1}{2} \cdot \frac{2m+3}{2} \cdot \frac{e^2}{2!} \right)^2 + \left(\frac{2m+1}{2} \cdot \frac{2m+3}{2} \cdot \frac{2m+5}{2} \cdot \frac{e^3}{3!} \right)^2 + \dots \right] \\ &+ 2 \left[\frac{2m+1}{2} \cdot e + \left(\frac{2m+1}{2} \right)^2 \cdot \frac{2m+3}{2} \cdot \frac{e^3}{2!} + \left(\frac{2m+1}{2} \cdot \frac{2m+3}{2} \right)^2 \cdot \frac{2m+5}{2} \cdot \frac{e^5}{2! \cdot 3!} \right. \\ &+ \dots \left. \right] \left[\frac{s+s^{-1}}{2} \right] + 2 \left[\frac{2m+1}{2} \cdot \frac{2m+3}{2} \cdot \frac{e^2}{2!} + \left(\frac{2m+1}{2} \right)^2 \cdot \frac{2m+3}{2} \cdot \frac{2m+5}{2} \cdot \frac{e^4}{3!} \right. \\ &+ \left. \left(\frac{2m+1}{2} \cdot \frac{2m+3}{2} \right)^2 \cdot \frac{2m+5}{2} \cdot \frac{2m+7}{2} \cdot \frac{e^6}{2! \cdot 4!} + \dots \right] \left[\frac{e^2 + e^{-2}}{2} \right] + \dots (A). \\ &\therefore (1 + e^2 - 2e \cos \varphi)^{-\frac{1}{2}(2m+1)} = \frac{1}{2} P_0 + P_1 \cos \varphi + P_2 \cos 2\varphi + P_3 \cos 3\varphi + \dots \\ &+ P_n \cos n\varphi. \end{aligned}$$

When $m=0, 1, 2, 3$, etc., we get

$$(1 + e^2 - 2e \cos \varphi)^{-\frac{1}{2}} = \frac{1}{2} A_0 + A_1 \cos \varphi + A_2 \cos 2\varphi + A_3 \cos 3\varphi + \dots \dots \dots (B).$$

$$(1 + e^2 - 2e \cos \varphi)^{-\frac{3}{2}} = \frac{1}{2} B_0 + B_1 \cos \varphi + B_2 \cos 2\varphi + B_3 \cos 3\varphi + \dots \dots \dots (C).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}}=\frac{1}{2}C_0+C_1\cos\varphi+C_2\cos2\varphi+C_3\cos3\varphi+\dots\dots\dots(D).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}}=\frac{1}{2}D_0+D_1\cos\varphi+D_2\cos2\varphi+D_3\cos3\varphi+\dots\dots\dots(E).$$

Let $\sin(\theta-\varphi)=e\sin\theta$, then $\tan\theta=\sin\theta/(\cos\varphi-e)$.

$$\therefore \cos(\theta-\varphi)\left(1-\frac{d\varphi}{d\theta}\right)=e\cos\theta.$$

$$\therefore \frac{d\varphi}{d\theta}=\frac{\cos(\theta-\varphi)-e\cos\theta}{\cos(\theta-\varphi)}=\frac{\sin\theta\sin\varphi+\cos\theta(\cos\varphi-e)}{\sqrt{1-e^2\sin^2\theta}}.$$

$$\therefore \frac{d\varphi}{d\theta}=\frac{(\cos^2\theta\operatorname{cosec}\theta+\sin\theta)\sin\varphi}{\sqrt{1-e^2\sin^2\theta}}=\frac{\sin\varphi\operatorname{cosec}\theta}{\sqrt{1-e^2\sin^2\theta}}$$

$$=\sqrt{\frac{(\cos\varphi-e)^2+\sin^2\varphi}{1-e^2\sin^2\theta}}=\sqrt{\frac{1+e^2-2e\cos\varphi}{1-e^2\sin^2\theta}}\dots\dots\dots(1_0).$$

$$\text{Also } \sqrt{1+e^2-2e\cos\varphi}=\sqrt{1-e^2\sin^2\theta}-e\cos\theta.$$

$$\therefore \cos\varphi=e\sin^2\theta+\cos\theta\sqrt{1-e^2\sin^2\theta}\dots\dots\dots(2_0).$$

$$\cos2\varphi=4e^2\sin^4\theta+4e\sin^2\theta\cos\theta\sqrt{1-e^2\sin^2\theta}+1-2(1+e^2)\sin^2\theta\dots\dots\dots(3_0).$$

$$\cos3\varphi=4e^3\cos^6\theta+12e^2\sin^4\theta\cos\theta\sqrt{1-e^2\sin^2\theta}+12e\sin^2\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$+4\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}-3e\sin^2\theta-3\cos\theta\sqrt{1-e^2\sin^2\theta}\dots\dots\dots(4_0).$$

$$\cos4\varphi=1-8e^2\sin^4\theta-16e\sin^2\theta\cos\theta\sqrt{1-e^2\sin^2\theta}-8\cos^2\theta(1-e^2\sin^2\theta)$$

$$+8e^4\sin^8\theta+32e^3\sin^6\theta\cos\theta\sqrt{1-e^2\sin^2\theta}+48e^2\sin^4\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$+8\cos^4\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}+32e\sin^2\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}\dots\dots\dots(5_0).$$

$$\cos5\varphi=16e^5\sin^{10}\theta+80e^4\sin^8\theta\cos\theta\sqrt{1-e^2\sin^2\theta}+160e^3\sin^6\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$+160e^2\sin^4\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}+80e\sin^2\theta\cos^4\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}$$

$$+16\cos^5\theta(1-e^2\sin^2\theta)^{\frac{5}{2}}-20e^3\sin^6\theta-60e^2\sin^4\theta\cos\theta\sqrt{1-e^2\sin^2\theta}$$

$$-60\sin^2\theta\cos^2\theta(1-e^2\sin^2\theta)-20\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}$$

$$+5e\sin^2\theta+5\cos\theta\sqrt{1-e^2\sin^2\theta}\dots\dots\dots(6_0),$$

$$\begin{aligned}
\cos 6\varphi = & 32e^6 \sin^2 \theta + 192e^5 \sin^4 \theta \cos \theta \sqrt{1-e^2 \sin^2 \theta} + 480e^4 \sin^6 \theta \cos^2 \theta (1-e^2 \sin^2 \theta) \\
& + 640e^3 \sin^8 \theta \cos^3 \theta (1-e^2 \sin^2 \theta)^{\frac{3}{2}} + 480e^2 \sin^4 \theta \cos^4 \theta (1-e^2 \sin^2 \theta)^2 \\
& + 192e \sin^2 \theta \cos^6 \theta (1-e^2 \sin^2 \theta)^{\frac{5}{2}} + 32 \cos^8 \theta (1-e^2 \sin^2 \theta)^3 - 48e^4 \sin^8 \theta \\
& - 192e^3 \sin^6 \theta \cos \theta \sqrt{1-e^2 \sin^2 \theta} - 288e^2 \sin^4 \theta \cos^2 \theta (1-e^2 \sin^2 \theta) \\
& - 192e \sin^2 \theta \cos^3 \theta (1-e^2 \sin^2 \theta)^{\frac{3}{2}} - 48 \cos^4 \theta (1-e^2 \sin^2 \theta)^2 + 18e^2 \sin^4 \theta \\
& + 18 \cos^2 \theta (1-e^2 \sin^2 \theta) + 36e \sin^2 \theta \cos \theta \sqrt{1-e^2 \sin^2 \theta} - 1 \dots \dots \dots (7_0).
\end{aligned}$$

Writing (B) in the following form :

$$(1+e^2-es-es^{-1})^{-\frac{1}{2}} = \frac{1}{2}A_0 + \frac{1}{2}A_1(s+s^{-1}) + \dots + \frac{1}{2}A_n(s^n+s^{-n}) + \dots \dots \dots (8_0).$$

Differentiating (8₀) we get,

$$\begin{aligned}
e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{3}{2}} = & A_1(1-s^{-2}) + 2A_2(s-s^{-3}) \\
& + 3A_3(s^2-s^{-4}) + \dots + nA_n(s^{n-1}-s^{-(n+1)}) + \dots \dots (9_0).
\end{aligned}$$

From (8₀) and (9₀) we get,

$$\begin{aligned}
\frac{1}{2}e(1-s^{-2})[A_0 + A_1(s+s^{-1}) + A_2(s^2+s^{-2}) + \dots + A_n(s^n+s^{-n}) + \dots] \\
= (1+e^2-es-es^{-1})[A_1(1-s^{-2}) + 2A_2(s-s^{-3}) + 3A_3(s^2-s^{-4}) \\
+ \dots + nA_n(s^{n-1}-s^{-(n+1)}) + \dots].
\end{aligned}$$

Equating coefficients of s^n we get,

$$\frac{1}{2}e(A_n - A_{n+2}) = (n+1)(1+e^2)A_{n+1} - e[nA_n + (n+2)A_{n+2}].$$

$$\therefore A_{n+2} = \frac{2(n+1)}{2n+3} \cdot \frac{1+e^2}{e} A_{n+1} - \frac{2n+1}{2n+3} A_n \dots \dots \dots (10_0).$$

\therefore When A_n and A_{n+1} are known we can easily find A_{n+2} .

Multiplying (B) by $\cos \varphi$ we get,

$$\frac{\cos \varphi}{(1+e^2-2e \cos \varphi)^{\frac{1}{2}}} = \frac{1}{2}A_0 \cos \varphi + \frac{1}{2}A_1(1+\cos 2\varphi) + \frac{1}{2}A_2(\cos \varphi + \cos 3\varphi) + \dots (11_0).$$

$$\text{Also } \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = F(e, \tfrac{1}{2}\pi) \dots \dots \dots (1).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta} d\theta = E(e, \tfrac{1}{2}\pi) \dots \dots \dots (2).$$

Integrating both sides of (B) and (11₀) between the limits 2π and 0, we get with the aid of (1₀), (2₀), (1), (2), the following :

$$\pi A_0 = \int_0^{2\pi} \frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4 \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = 4F(e, \tfrac{1}{2}\pi).$$

$$\therefore A_0 = (4/\pi)F(e, \tfrac{1}{2}\pi) \dots \dots \dots (12_0).$$

$$\begin{aligned} \pi A_1 &= \int_0^{2\pi} \frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4e \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{1-e^2\sin^2\theta}} + \int_0^{2\pi} \cos\theta d\theta \\ &= \frac{4}{e} \left[\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} - \int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta} d\theta \right] = \frac{4}{e} [F(e, \tfrac{1}{2}\pi) - E(e, \tfrac{1}{2}\pi)]. \end{aligned}$$

$$\therefore A_1 = (4/\pi e) [F(e, \tfrac{1}{2}\pi) - E(e, \tfrac{1}{2}\pi)].$$

[To be Continued.]

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

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[Continued from October Number.]

CHAPTER III.

MULTIPLICATION OF EXTENSIVE QUANTITIES. DIFFERENT KINDS OF MULTIPLICATION.

22. In the multiplication of extensive quantities expressed in terms of *units*, it is assumed that the distributive law holds, and that numerical coefficients may be treated as in elementary algebra (16).